# Section 14.2

## Limits and Continuity in Several Variables





Prelecture Review Video

# 1 Limits of 2-Variable Functions



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# Limits of Functions of Two Variables

Let $f(x, y)$ be a function of two variables. The statement

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

means that:

- The closer a point $(x, y)$ gets to $(a, b)$, the closer $f(x, y)$ gets to $L$.
- As $\|(x, y) - (a, b)\|$ approaches 0, so does $f(x, y) - L$.
- We can make the value of $f(x, y)$ as close to $L$ as we like by requiring that $(x, y)$ is sufficiently close to $(a, b)$.

Multivariable limits are more complex than single-variable limits because $(x, y)$ can approach $(a, b)$ from many possible directions (not just two).

▸ Video

# Limits of Functions of Two Variables

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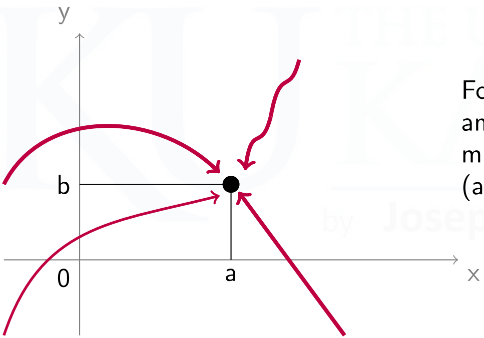


For the limit of $f$ to exist at $(a, b)$ and equal $L$, the values of $f(x, y)$ must approach $L$ on **all** these curves (and infinitely many others).

That is, $\lim_{(x,y)\to(a,b)} f(x, y) = L$ means that $\lim_{t\to c} f(x(t), y(t)) = L$

for **all** parametric curves $(x(t), y(t))$ for which $\lim_{t\to c}(x(t), y(t)) = (a, b)$

# Limits and Continuity in Two Variables

The basic limit laws (Rogawski, §2.3) still apply to limits in two variables:

**Sum Law**
**Product Law**
**Power/Root Laws**
**Constant Multiple Law**
**Quotient Law**

- We say that $z = f(x, y)$ is **continuous at a point $(a, b)$** if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

- We say that $z = f(x, y)$ is **continuous on a domain $D$** if it is continuous at every point in $D$.

# 2 Continuity of Elementary Functions, Computing Limits in the Domain

# Continuity of Elementary Functions

Just like in Calculus I, there is good news:

**Elementary functions are continuous on their domains!**

That is, you use Direct Substitution when functions are elementary. (An **elementary function** is one that can be constructed from building blocks like polynomials, rational functions, root functions, exponential, logarithmic, trigonometric, and inverse-trig functions, using arithmetic operations and function composition.)

**Example 1:** $$\lim_{(x,y)\to(-2,1)} \frac{2x^2}{4x+y} = -\frac{8}{7}$$

**Example 2:** $$\lim_{(x,y)\to(2,4)} (y\arcsin(x/y))^x = (4\arcsin(1/2))^2 = 4(\pi)^2/9$$

Video

# 3 Examples of Indeterminate Forms, When the Limit Doesn't Exist

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# Determining Limits of Two-Variable Functions

How do we calculate limits when we cannot use direct substitution?

**Example 3:** Let $f(x,y) = \dfrac{xy}{x^2+y^2}$. Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$.

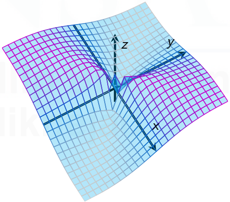
Solution: For the limit to exist, the function $f(x,y)$ must approach the same value along **all** curves approaching $(0,0)$. *If any two curves disagree, then the limit does not exist.*

- As $(x,y) \to (0,0)$ along $y = 0$:

$$\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{0}{x^2+0} = 0$$

- As $(x,y) \to (0,0)$ along $y = x$:

$$\lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{x^2}{x^2+x^2} = \frac{1}{2}$$

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**The limit does not exist!**

# Determining Limits of Two-Variable Functions

**General principles for determining limits:**

- In order for $\lim_{(x,y)\to(a,b)} f(x,y)$ to equal $L$, the function $f(x,y)$ must approach $L$ along **all** paths approaching $(a,b)$.
- If any two curves disagree, then the limit does not exist.

**Warning:** You **cannot** show that a limit exists by showing that two specific curves agree (because there could be a third one that disagrees).

Even showing that infinitely many curves agree might not be enough!

# Determining Two-Variable Limits: Be Careful!

**Example 4:** Let $g(x,y) = \dfrac{x^2}{x^2+y^2}$. Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} g(x,y)$.

Solution:

- Along $x = y$:

  $$\lim_{x\to 0} g(x,x) = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}.$$

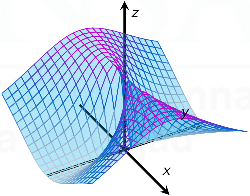
- Along $x = -y$: $\displaystyle\lim_{x\to 0} g(x,-x) = \frac{1}{2}$.

  **But**. . .

- Along $y = 0$:

  $$\lim_{x\to 0} g(x,0) = \lim_{x\to 0} \frac{x^2}{x^2} = 1.$$

  **The limit does not exist!**

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# Determining Two-Variable Limits: Be Careful!

**Warning:** Even showing that infinitely many curves agree may not be enough to conclude that a limit exists!

**Example 5:** Let $g(x,y) = \dfrac{x^2 y}{x^4 + y^2}$. Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} g(x,y)$. Link

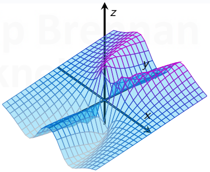
Solution: We cannot plug in $(x,y) = (0,0)$ since it is not in the domain.

- Along $y = mx$ for any number $m$:

$$\lim_{x\to 0} g(x, mx) = \lim_{x\to 0} \frac{mx^3}{x^4 + m^2x^2} = 0$$

- Along $y = x^2$:

$$\lim_{x\to 0} g(x, x^2) = \lim_{x\to 0} \frac{x^4}{x^4 + x^4} = \frac{1}{2}$$

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**The limit does not exist!**

# 4 Examples of Indeterminate Forms, When the Limit Exists



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# So What Do We Do Instead?

This is very frustrating. Checking *all* paths approaching $(a, b)$ would be an impossible task.

Fortunately, there is one powerful trick: **converting a limit to polar coordinates.**

$$\lim_{(x,y)\to(0,0)} f(x,y) \;=\; \lim_{\substack{r\to 0^+ \\ \theta\to ??}} f(r\cos(\theta), r\sin(\theta))$$

If we can show that the value of $\theta$ does not affect the limit, we have transformed the problem into calculating one-variable limit involving $r$.

Typically, we need the Squeeze Theorem to show that $\theta$ does not matter.

Video

**Example 6:** Let $j(x,y) = \dfrac{e^{x^2+y^2}-1}{x^2+y^2}$. Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} j(x,y)$.

Solution: To save time, we will attempt to show that the limit exists and skip checking individual paths. Converting to polar coordinates gives

$$\lim_{(x,y)\to(0,0)} \frac{e^{x^2+y^2}-1}{x^2+y^2} = \lim_{\substack{r\to 0^+ \\ \theta\to ??}} \frac{e^{r^2}-1}{r^2} = \lim_{r\to 0^+} \frac{e^{r^2}-1}{r^2}.$$

We can drop the "$\theta \to$ **??**" because $\theta$ has disappeared. This is an 0/0 form, so we can apply L'Hôpital's Rule:

$$\lim_{r\to 0^+} \frac{2re^{r^2}}{2r} = \lim_{r\to 0^+} e^{r^2} = e^0 = 1.$$

**Example 7:** Let $h(x,y) = \dfrac{x^2 y}{x^2+y^2}$. Evaluate $\lim\limits_{(x,y)\to(0,0)} h(x,y)$.

Solution: Converting to polar coordinates gives

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = \lim_{\substack{r\to 0^+ \\ \theta\to ??}} \frac{(r^2\cos^2(\theta)(r\sin(\theta))}{r^2} = \lim_{\substack{r\to 0^+ \\ \theta\to ??}} r\cos^2(\theta)\sin(\theta).$$

Observe that outputs of $\cos(\theta)$ and $\sin(\theta)$ are both in $[-1,1]$ for all real $\theta$, so $\cos^2(\theta)\sin(\theta)$ is as well.

Therefore, $-r \leq r\cos^2(\theta)\sin(\theta) \leq r$, so

$$\underbrace{\lim_{r\to 0^+} -r}_{=0} \quad \leq \quad \lim_{\substack{r\to 0^+ \\ \theta\to ??}} r\cos^2(\theta)\sin(\theta) \quad \leq \quad \underbrace{\lim_{r\to 0^+} r}_{=0}$$

and our limit equals 0 by the Squeeze Theorem.

**Example 8:** Let $k(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$. Evaluate $\lim_{(x,y)\to(0,0)} k(x, y)$.

Solution: Rewrite the limit in polar coordinates.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{\substack{r\to 0^+ \\ \theta\to ??}} \frac{r^2(\cos^2(\theta) - \sin^2(\theta))}{r^2} = \lim_{\substack{r\to 0^+ \\ \theta\to ??}} \cos(2\theta)$$

But this very clearly depends on $\theta$, which says literally that the value of $k(x, y)$ approaches different values along different paths to the origin.

Therefore, **the original limit does not exist.**